DCS/CSCI 2350: Social & Economic Networks

How are networks formed in real world?

Modeling Networks

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Reading

- Newman's *Networks,* Ch 12 (Canvas)
  - Erdos-Renyi random graphs
- Selected topics: Chapters 1, 4, 5 of Jackson's book (Canvas)
  - Watts-Strogatz and preferential attachment
- Optional: Chapters 3, 4 of Watts's Six Degrees book (for behind the scene)

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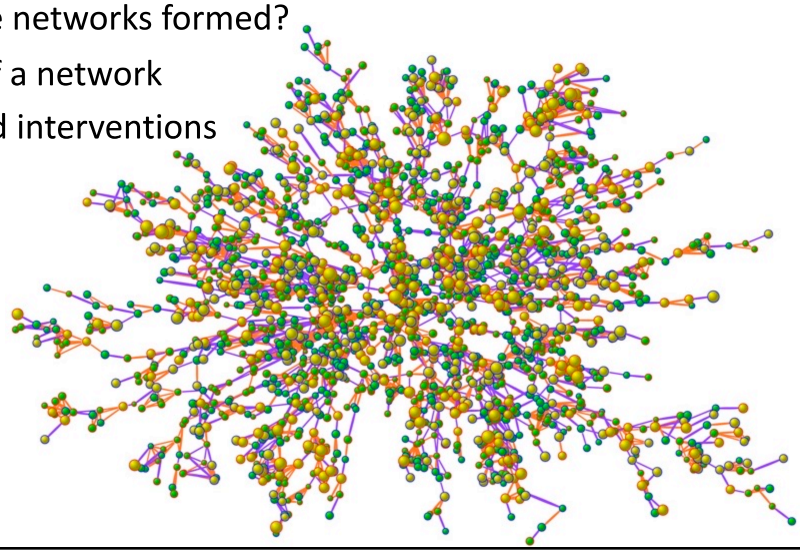
Why model networks?

- How are networks formed?
- Effect of a network
- Targeted interventions



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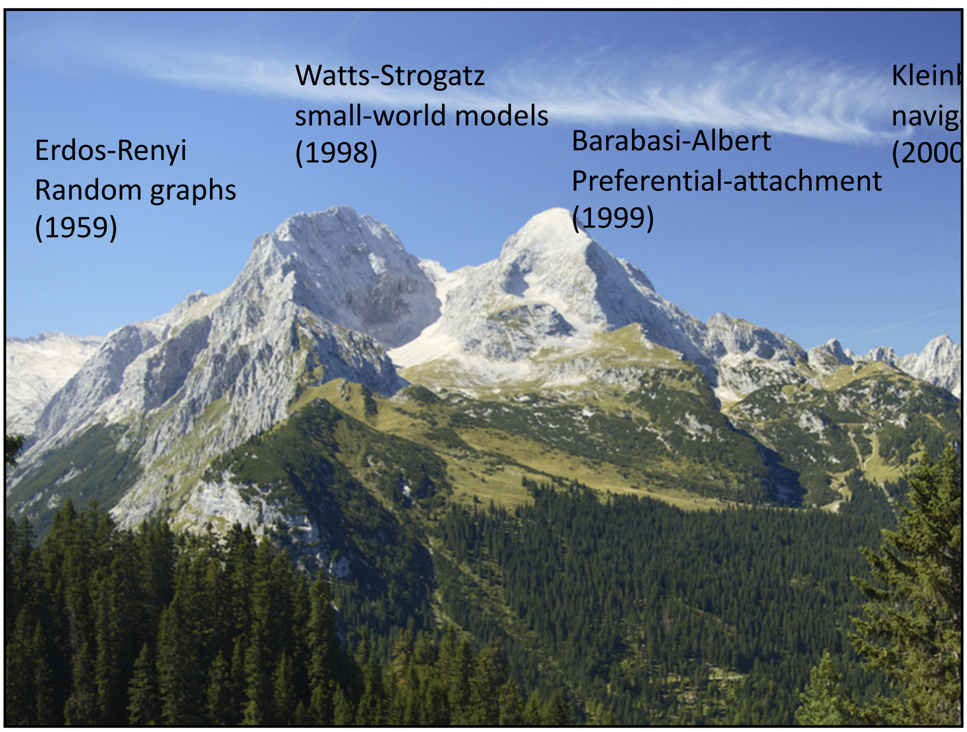

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Why random network models?

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Why randomization?

- First attempt -> richer models
- Powerful!
  - An ensemble of networks, not just one network
  - Can capture real-world network properties
- Context-free
- Computationally tractable

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Prelude

- Power-law degree distribution, $p_k = C\, k^{-\alpha}$
- (Puzzle 2)
  Max possible # of edges = n(n-1)/2 = ${}^{n}C_2$
- $\binom{n}{k} = {}^{n}C_k = n! / (k!(n-k)!)$

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Erdos-Renyi random graphs (or random graphs)

- Static
  - Given n nodes (constant)
- Variant I
  - Inputs: number of nodes n and number of edges m
  - Create m edges uniformly at random out of ${}^{n}C_2$ total possible edges
- **Variant II**
  - Inputs: number of nodes n and probability of forming an edge = p

(Puzzle 2)
Max possible # of edges = n(n-1)/2 = ${}^{n}C_2$

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Properties of Erdos-Renyi graphs

- Every simple graph is possible!
- How can we say something regarding properties?
  1. Estimate the probability of a property
  2. Limiting behavior: n → infinity

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Properties of Erdos-Renyi graphs

- Degree distribution
- Clustering coefficient
- Small-world effect
- Giant component

> ▶ Power-law degree distribution, $p_k = C\, k^{-\alpha}$
>
> ▶ (Puzzle 2) Max possible # of edges = $n(n-1)/2 = {}^nC_2$
>
> ▶ $\binom{n}{k} = {}^nC_k = {}^nC_k = n!/[k!(n-k)!]$

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Degree distribution

- $p_k = e^{-c} c^k / k!$ [Poisson distribution]
- Here, mean degree $c = p(n-1)$
  [AKA average deg. or expected deg.]

> - Power-law degree distribution, $p_k = C k^{-\alpha}$
> - (Puzzle 2) Max possible # of edges = $n(n-1)/2 = {}^nC_2$
> - $\binom{n}{k} = {}^nC_k = {}^nC_k = n!/[k!(n-k)!]$

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Derivation (optional)

$$p_k = \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

$$= \frac{(n-1)!}{(n-1-k)!\,k!} p^k (1-p)^{n-1-k}$$

$$= \frac{(n-1)(n-2)\cdots(n-k)\,(n-k-1)!}{(n-1-k)!\,k!} p^k (1-p)^{n-1-k}$$

$$\simeq \frac{(n-1)^k p^k}{k!} (1-p)^{n-1-k}$$

$$= \frac{c^k}{k!} (1-p)^{n-1-k}$$

$$\simeq \frac{c^k e^{-c}}{k!}.$$

$$\ln (1-p)^{n-1-k} = (n-1-k) \ln (1-p)$$

approx

$$= -p(n-1-k)$$

$$\simeq -p(n-1)$$

$$= -c$$

$$\Rightarrow (1-p)^{n-1-k} \simeq e^{-c}$$

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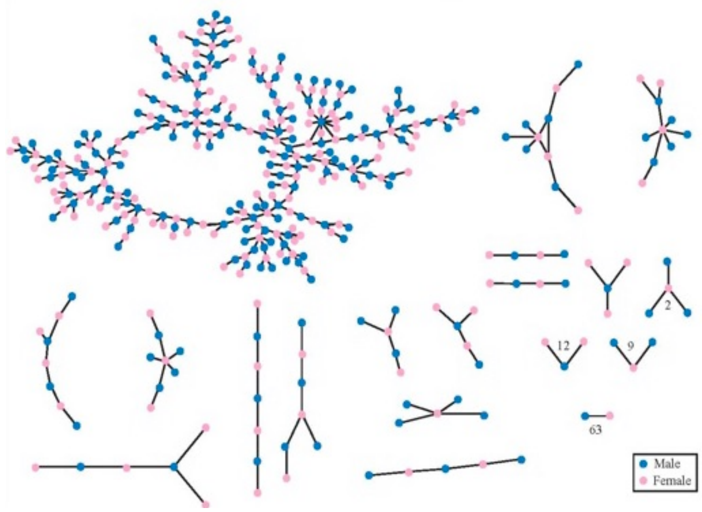
High-school relationships (Bearman et al, 2004)



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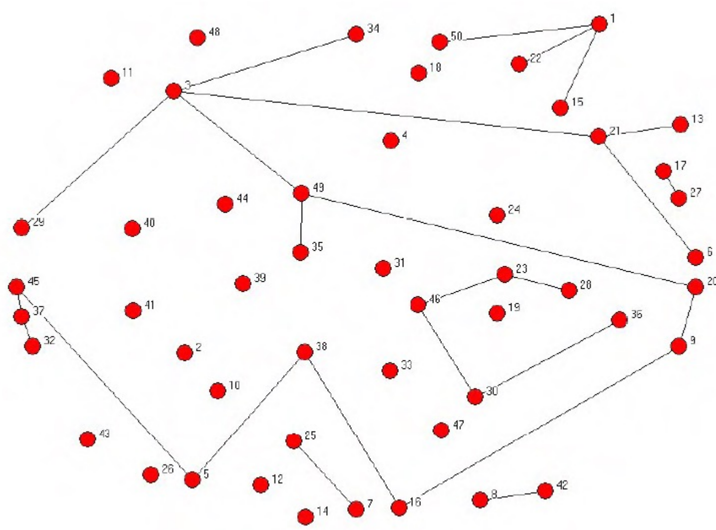
Random graph with p = 0.02



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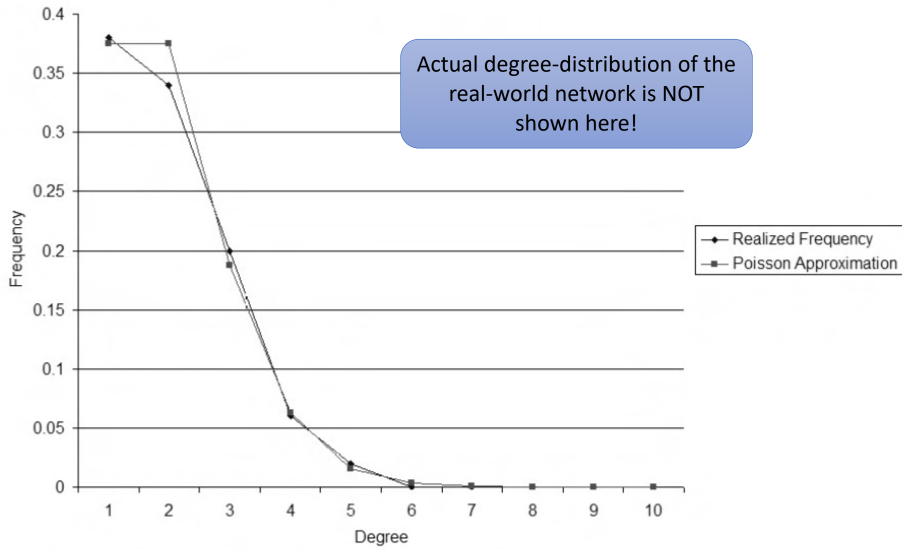
Degree distribution: p = 0.02

Approximation by Poisson distribution



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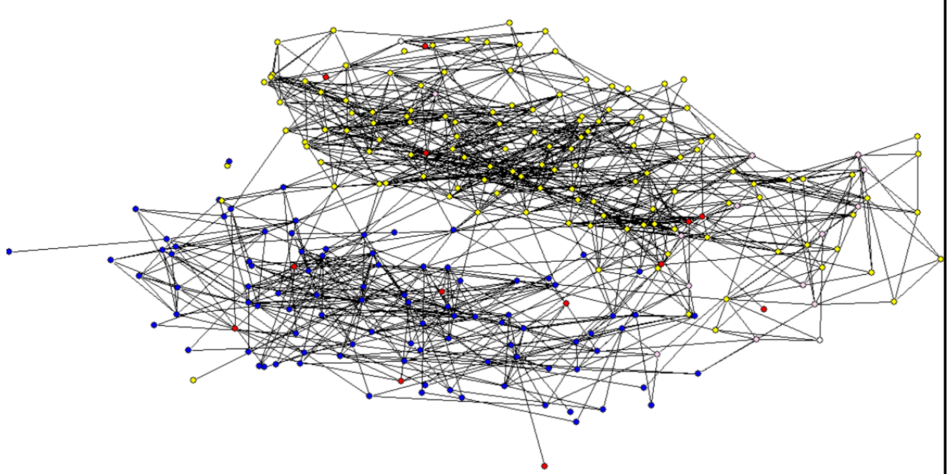
High-school friendships (Currarini et al, 2007)



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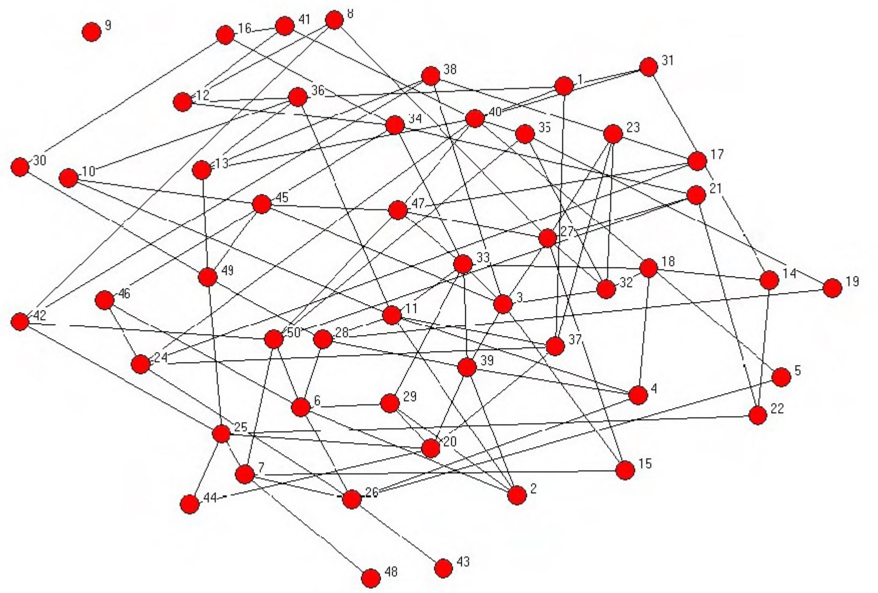
Random graph with p = 0.08



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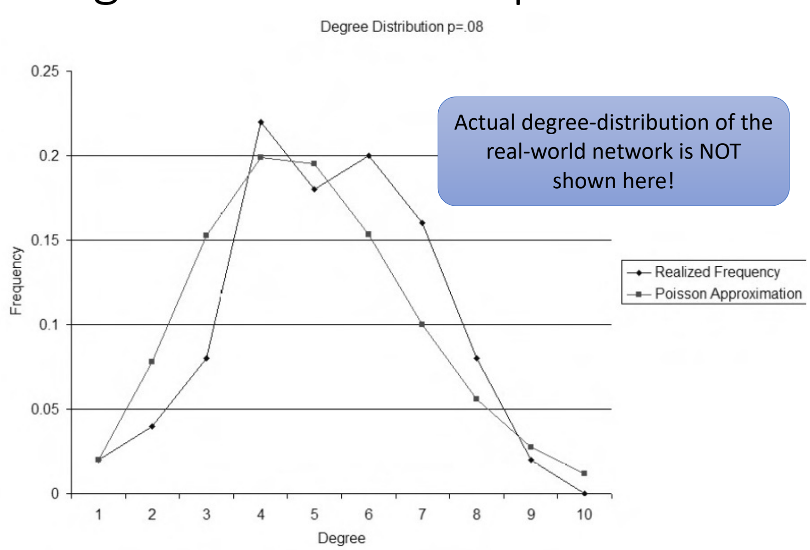
Degree distribution: p = 0.08



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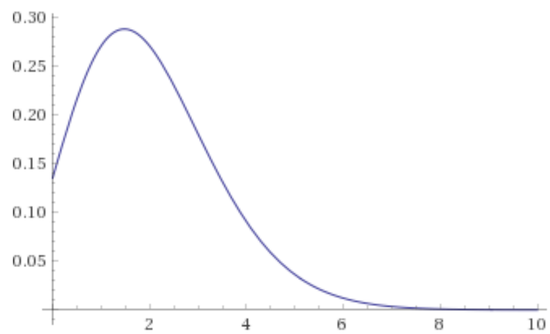
Plotting Erdos-Renyi degree distribution

WolframAlpha computational intelligence.

plot exp(-2)*2^k/k! for k = 0 to 10

- Plug in Poisson distribution
- Expected degree, c = 2



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Erdos-Renyi:
Giant component

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Phase transition & giant comp (GC)

Let q fraction of nodes be in the GC:

Fraction of nodes outside of the GC = 1-q

Prob of finding a node outside of the GC irrespective of its degree = right hand side below

$$1-q = \sum_k p_k (1-q)^k$$

$$1-q = \sum_k \frac{e^{-c} c^k}{k!} (1-q)^k$$

$$= e^{-c} \sum_k \frac{[c(1-q)]^k}{k!}$$

$$= e^{-c} e^{c(1-q)}$$

$$= e^{-cq}$$

$$1-q = e^{-cq}$$

$$\boxed{q = 1 - e^{-cq}}$$

Experimental solution:

GC emerges when c > 1

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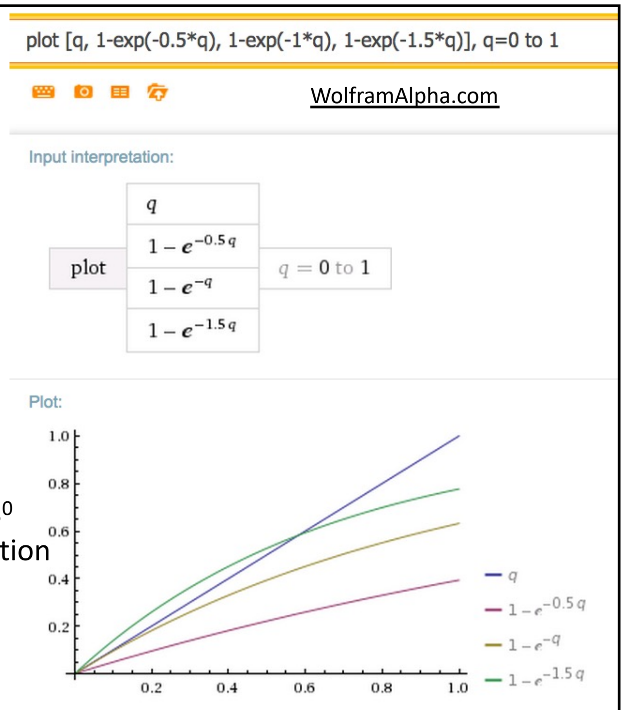
Phase transition & giant comp.



- $q = 1 - e^{-cq}$
  - X-axis is q
  - Y-axis is $1 - e^{-cq}$
  - Intersection with 45^0 line solves the equation
- Giant component emerges when c > 1

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Giant component: Netlogo experiments

1. Prof. Irfan’s program: https://mtirfan.com/Erdos-Renyi.html

2. Netlogo -> Models Library -> Networks -> Giant Component

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Erdos-Renyi:
Clustering coefficient

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Erdos-Renyi:
Small-world property

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Properties of Erdos-Renyi graphs

- Degree distribution 
- Giant component 
- Clustering coefficient 
- Small-world effect 

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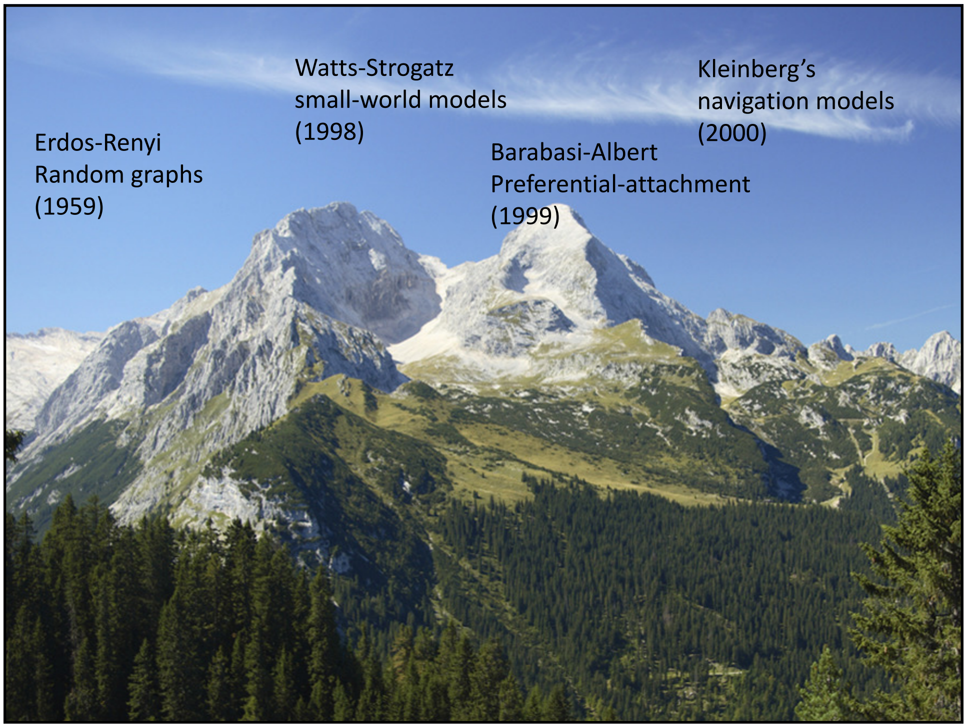

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Question

- How to create random graphs that capture the real-world clustering properties?

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~~dom graph limit~~. For this simple model, one surprising result is that on average, the first five random rewirings reduce the average path length of the network by one-half, *regardless of the size of the network*. ~~The big-~~

Duncan Watts, *Six Degrees*, pg. 89

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~~dom graph limit~~. For this simple model, one surprising result is that on average, the first five random rewirings reduce the average path length of the network by one-half, *regardless of the size of the network*. The bigger the network, the greater the effect of each individual random link so the impact of adding links becomes effectively independent of size. The law of diminishing returns, however, is just as striking. A further 50 percent reduction (so that now the average path length is at one-fourth of its original value) requires roughly another fifty links—roughly ten times as many as for the first reduction and for only half as much over-

Duncan Watts, *Six Degrees*, pg. 89

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Watts-Strogatz small-world model (1998)

- Degree distribution 
- Clustering coefficient 
- Giant component 
- Small-world effect 

Experiment with NetLogo:
File → Models Library →
Networks → Small Worlds

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Question

- How to create random graphs that capture the real-world degree-distribution?

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Examples

- Pareto (1890s)
  - Wealth distribution, city sizes
- Herbert Simon (1955):
  - System grows over time with new objects entering
  - Existing objects grow proportional to their size
  - "The rich gets richer faster than the poor"
- Derek Price (1965)
  - Citation network: # of citations of a paper is proportional to the # of citations it has

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Barabasi-Albert Preferential-attachment model (1999)

- Nodes are born over time (only one node at a time). DOB: {0, 1, 2, ..., t, ...}
- Degree of node i at time t: $d_i(t)$
- Upon birth, a node forms M edges
  - Pr(attaching to node i) is proportional to $d_i(t)$

M is the only model parameter!

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Preferential attachment

Degree distribution is power law!

(derivation)

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Barabasi-Albert Preferential-attachment model (1999)

- Degree distribution 
- Clustering coefficient ?
- Giant component 
- Small-world effect 

Experiment with NetLogo:
File → Models Library →
Networks → Preferential Att.

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